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CERTAIN PROBLEMS OF MOTION OF INCOMPRESSIBLE VISCO-FRIABLE MATERIALS*

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The steady motion of an incompressible visco-friable medium is considered. Equations of motion in the boundary layer are obtained and the flow over a flat plate is investigated. The flow in a plane diffuser and between rotating cylinders is analyzed.

Models of visco-friable continuous medium were proposed in a number of publications /1—3/. A visco-friable material is essentially viscoplastic but with its yield stress dependent on pressure, i.e. the deviator of the stress tensor is defined by the expression /3/

$$\sigma_{ik} = \left(\frac{\sqrt{2}\tau_s}{\sqrt{\varepsilon_{ab}\varepsilon_{ab}}} + 2\eta\right)\varepsilon_{ik}, \quad \tau_s = kp, \quad \varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \tag{1}$$

where k is the coefficient of internal pressure, η is the viscosity coefficient, p is the pressure which is assumed positive to make it possible to consider the visco-friable medium as a continuous one.

The presence in visco-friable media of limit shear stress results in the formation of zones where such medium moves like a solid body. The position and shape of boundaries that separate viscous flow zones from those of quasisolid flow have to be determined in the process of solving the problem. Conditions at the solid zone boundary were determined for the conventional viscoplastic material in integral form in /4/ and in final form in /5/.

Below, we present examples of steady motion of incompressible visco-friable materials for which Eq. (1) is valid, and whose viscosity coefficient η is assumed constant.

Let us consider the equations of a steady plane boundary layer in the absence of mass forces of a visco-friable medium defined by the equations of motion of a continuous medium in terms of stresses and by Eq.(1).

We introduce the dimensionless quantities

$$\mathrm{Re} = \frac{\rho UL}{\eta}, \quad \mathrm{Bi} = \frac{kp_0L}{\eta U}, \quad x = \frac{x_1}{L}, \quad y = \frac{x_2}{\varepsilon L}, \quad v_x = \frac{v_1}{U}, \quad v_y = \frac{v_2}{\varepsilon U}, \quad p^* = \frac{p}{p_0}$$

where Re is the Reynolds number, Bi is a dimensionless parameter that defines the ratio of plastic and viscous energy dissipation, x and y are the longitudinal and transverse coordinates, p_0 is the characteristic pressure, and ε is a small, so far undefined, quantity dependent on Re and Bi.

Let us assume that the quantity $\partial v_x/\partial y$ is positive throughout the boundary layer, and disregard terms of higher order with respect to ε. Then from the equations of motion we obtain the following equations:

$$\varepsilon^2\,\mathrm{Re}\left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\varepsilon^2\frac{p_0L}{\eta U}\frac{\partial p^*}{\partial x} + \frac{\partial^2 v_x}{\partial y^2} + \varepsilon\,\mathrm{Bi}\,\frac{\partial p^*}{\partial y}, \quad 0 = -\varepsilon\frac{p_0L}{\eta U}\frac{\partial p^*}{\partial y} + \varepsilon^2\,\mathrm{Bi}\left(\frac{\partial p^*}{\partial x} - 2\frac{\partial}{\partial y}\left(p^*\frac{\partial v_x/\partial x}{\partial v_x/\partial y}\right)\right)$$

Let $kp_0 \gg \rho U^2$. We then have the plastic boundary layer ($\varepsilon = \mathrm{Bi}^{1/2}$) whose equation is

$$0 = -\frac{\partial p^*}{\partial x} + k\frac{\partial^2 v_x}{\partial y^2} + k^2\left(\frac{\partial p^*}{\partial x} - 2\frac{\partial}{\partial y}\left(p^*\frac{\partial v_x/\partial x}{\partial v_x/\partial y}\right)\right), \quad 0 = \frac{\partial p^*}{\partial y}, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{2}$$

In this case the boundary layer thickness δ grows with increasing flow velocity $\delta \sim L[\eta U/(kp_0L)]^{1/2}$. Note that for a viscoplastic medium ($\tau_s = \text{const}$) $\delta \sim L[\eta U/(\tau_sL)]^{1/3}$.

Let us apply Eq.(2) in the problem of flow over a semi-infinite plane plate of a plane parallel stream of visco-friable medium. Let the plane of the plate coincide with the half-plane x_1x_3 which corresponds to positive x_1 and the line $x_1 = 0$ represents the plate leading edge. Reverting in Eqs.(2) to dimensional quantities, we obtain

$$0 = \eta\frac{\partial^2 v_1}{\partial x_2^2} - 2k^2p_0\frac{\partial}{\partial x_2}\left(\frac{\partial v_1/\partial x_1}{\partial v_1/\partial x_2}\right), \quad p = p_0, \quad \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \tag{3}$$

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with boundary conditions

$$v_1 = v_2 = 0, \quad x_1 \geqslant 0, \quad x_2 = 0; \quad v_1 = U, \quad x_2 = \pm\infty$$

We seek a solution of the form

$$v_1 = Uf(\xi), \quad v_2 = U\sqrt{\frac{\eta U}{k p_0 x_1}}\, f_1(\xi), \quad \xi = x_2\sqrt{\frac{k p_0}{x_1 \eta U}} \tag{4}$$

where f and f_1 are some dimensionless functions. Solving Eqs.(3) with allowance for (4) we can obtain $f = -k\xi^2 + \sqrt{2k}\,|\xi|, \; |\xi| \leqslant \sqrt{2/k}$. The friction force acting on a unit area of the plate surface is

$$\sigma_{12} = k p_0 + \eta \frac{\partial v_1}{\partial x_2}\Big|_{x_2=0} = k p_0\left(1 + \sqrt{\frac{2\eta U}{p_0 x_1}}\right)$$

For a plate of length l (along the x_1-axis) the total friction force acting on per unit of plate width (in the direction of axis x_3) is

$$F = 2\int_0^l \sigma_{12}\, dx_1 = 2k p_0 l\,(1 + 2\sqrt{2k}\,\mathrm{Bi}^{-1/2}) \tag{5}$$

where friction on the two sides of the plate is taken into account by the factor 2.

Let us consider the steady motion of a visco-friable medium between two plane rough walls at an angle of 2α to each other, with the outflow taking place along the intersection line of the two planes.

We select cylindrical coordinates r, φ, z with the z-axis on the intersection line of the planes and angle φ measured from the bisectrix of the angle 2α. The motion is uniform along the z-axis, and it is reasonable to assume it to be purely radial, i.e. $v_\varphi = v_z = 0, v_r = v(r, \varphi)$. The equations of motion form the system

$$v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \left(\nu + \frac{\sqrt{2}kp}{2\rho A}\right)\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) + \frac{\partial v}{\partial r}\frac{\partial}{\partial r}\left(\frac{\sqrt{2}kp}{\rho A}\right) + \frac{1}{r^2}\frac{\partial v}{\partial \varphi}\frac{\partial}{\partial \varphi}\left(\frac{\sqrt{2}kp}{\rho A}\right) \tag{6}$$

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \frac{2}{r^2}\left(\nu + \frac{\sqrt{2}kp}{2\rho A}\right)\frac{\partial v}{\partial \varphi} + \frac{v}{r^2}\frac{\partial}{\partial \varphi}\left(\frac{\sqrt{2}kp}{\rho A}\right) + \frac{1}{2r}\frac{\partial v}{\partial \varphi}\frac{\partial}{\partial r}\left(\frac{\sqrt{2}kp}{\rho A}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) = 0; \quad A^2 = \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{v}{r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial v}{\partial \varphi}\right)^2$$

where the last equation shows that rv is a function of only φ. Hence we seek a solution of the form

$$v = \frac{\nu}{r}u(\varphi), \quad p = \rho\frac{\nu^2}{r^2}w(\varphi) \tag{7}$$

Substituting (7) into (6), we obtain for functions u and w the system of equations

$$\sqrt{2}u\frac{d}{d\varphi}\left(\frac{kw}{C}\right) + 2u'\left(1 + \frac{\sqrt{2}}{2}\frac{kw}{C}\right) - w' = 0, \quad 2w + u^2 + u''\left(1 + \frac{\sqrt{2}}{2}\frac{kw}{C}\right) + \frac{\sqrt{2}}{2}u'\frac{d}{d\varphi}\left(\frac{kw}{C}\right) = 0 \tag{8}$$

$$C = \sqrt{2u^2 + \frac{u'^2}{2}}, \quad u' = \frac{du}{d\varphi}, \quad w' = \frac{dw}{d\varphi}$$

The problem has been, thus, reduced to the integration of ordinary differential equations. The condition of velocity vanishing at the walls and the stipulation that the same quantity $Q > 0$ of friable medium

$$u(\pm\alpha) = 0, \quad Q = \rho\int_{-\alpha}^{\alpha} rv\, d\varphi = \rho\nu\int_{-\alpha}^{\alpha} u\, d\varphi$$

passes through any cross section $r = \mathrm{const}$ constitute the boundary conditions.

For the motion considered here the ratio $R = Q/(\rho\nu)$ represents the Reynolds number.

Let us assume that the motion is symmetric about the plane $\varphi = 0$ and that $u(\varphi)$ changes monotonically from zero at $\varphi = \pm\alpha$ to $u = u_0 > 0$ at $\varphi = 0$.

A symmetrically divergent flow of a viscous fluid in a diffuser is possible for a given angle of divergence only at Reynolds numbers not exceeding a certain limit. For a visco-friable medium the more stringent condition is that of positive pressure $(w > 0)$.

Regions of existence of visco-friable medium symmetric flows are shown in Fig.1 in the α, R plane for various values of the dry friction coefficient, i.e. of flows of which the condition $w > 0$ is satisfied. For a given α there exists an R_0 such that when $R > R_0$ the flow of a visco-friable medium is impossible without continuity breakdown. The dash line in

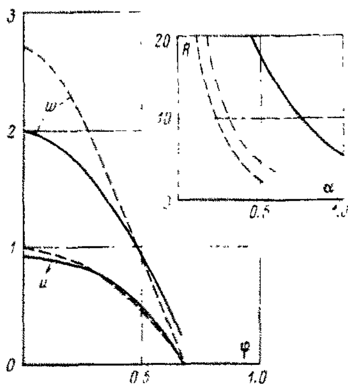

Fig.1

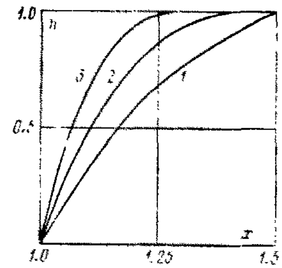

Fig.2

Fig.1 corresponds to $k = 0.3$, the dash-dot line to $k = 0$, and the solid line represents the boundary of the region of symmetric flow existence for a viscous fluid. The results of numerical integration of system (8) for $R = 0.88, \alpha = 0.67$ are also plotted in Fig.1, where the solid lines correspond to $k = 0$ and the dash lines to $k = 0.3$.

Let us consider the motion of a visco-friable material contained between two infinite coaxial cylinders rotating about their axis at angular velocities Ω_1 and Ω_2. The cylinder radii are R_1 and R_2 with $R_2 < R_1$. A similar problem ($\Omega_2 = 0$) was considered in /6/ without taking into account viscosity.

We select the cylindrical coordinates r, φ, z with the z-axis on the cylinder axis. Owing to symmetry we have $v_z = v_r = 0,\ v_\varphi = v(r),\ p = p(r)$.

The equations of motion in cylindrical coordinates reduce in this case to two equations

$$\nu\left(\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}\right)+\frac{k}{\rho}\left(\frac{dp}{dr}+2\frac{p}{r}\right)\operatorname{sgn}\left(\frac{dv}{dr}-\frac{v}{r}\right)=0,\quad \frac{dp}{dr}=\rho\frac{v^2}{r} \tag{9}$$

whose solution is sought in the form

$$v=\frac{\nu_0}{R_1}f(x),\quad p=\frac{\rho\nu_0^2}{R_1^2}g(x),\quad x=\frac{r}{R_1},\quad 1\leqslant x\leqslant\frac{R_2}{R_1},\quad \nu_0=\frac{\eta}{k\rho}$$

The general solution of system (9) depends on three arbitrary constants for whose determination it is necessary to know in addition to the cylinder rotation velocities, one more parameter, for instance, the pressure on one of the cylinders. Thus the boundary conditions for system (9) can be specified in the form

$$f(1)=\frac{\Omega_1R_1^2}{\nu_0},\quad f\left(\frac{R_2}{R_1}\right)=\frac{\Omega_2R_2R_1}{\nu_0},\quad g(1)=p_1\frac{R_1^2}{\rho\nu_0},\quad p_1=p(R_1)$$

When $\Omega_1 = \Omega_2 = \Omega$ the medium and the cylinders rotate as a single whole $v = \Omega r$. Let us consider the case when only the external cylinder rotates: $\Omega_1 = 0, \Omega_2 = \Omega$. The results of numerical integration of system (9) in the case of $R_2/R_1 = 1.5, \Omega R_1^2/\nu_0 = 1.0$ are shown in Fig.2, where the dimensionless radii $x = r/R_1$ are plotted on the axis of abscissas, and the dimensionless angular frequency $h = f/x$ appears on the axis of ordinates. Curves 1, 2, and 3 correspond to the following values of the dimensionless pressure $g(1)$: 0.5, 5.0, and 15.0, respectively, on the inner cylinder.

It will be seen that as the pressure is increased, regions of the visco-friable medium rigidity may appear in it. The expression for the moment of friction forces acting on the cylinders can be obtained only in the case of a norrow gap between the cylinders, when there are no rigidity regions. The moment of friction forces (per unit of cylinder length) in that approximation is

$$M=\frac{4\pi R_1^2R_2^2\eta\Omega}{R_2^2-R_1^2}\left(1+\frac{kp}{\eta\Omega}\ln\frac{R_2}{R_1}\right) \tag{10}$$

where p is the pressure in the medium which in the considered here approximation may be assumed constant. When $k = 0$ formula (10) reduces to the known formula for a viscous fluid.

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